The Product Rule

A Demonstration

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If we set $u = f(x)$ and $v = g(x)$, then this product rule can be written

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Product of two functions

Example

To find $f'(x)$ if $f(x) = 3x^2e^{x^2}$ begin by thinking of $f(x)$ as the product

$$f(x) = (3x^2)(e^{x^2}).$$

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Next, use the product rule

$$f'(x) = (6x)e^{x^2} + 3x^2(2xe^{x^2})$$

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Next, use the product rule and then factor to obtain

$$\begin{aligned} f'(x) &= \underbrace{(6x)e^{x^2}}_{\frac{du}{dx}v} + \underbrace{3x^2(2xe^{x^2})}_{u\frac{dv}{dx}} \\ &= 6xe^{x^2}(1+x^2). \end{aligned}$$

Proof of the product rule

From the definition of the derivative we have

$$D[f(x)g(x)] = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

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More details

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&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \lim_{h\to 0} g(x+h) \\
&\quad + f(x) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x)g(x) + f(x)g'(x).
\end{aligned}$$